

✓ B.Sc II (H)

Theorem : — If at the point (x, y) , the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ of the function $u = f(x, y)$ exist and are continuous, then the function is differentiable at that point.

Proof : — The condition that the functions $\frac{\partial u}{\partial x}$ and

$\frac{\partial u}{\partial y}$ are continuous implies that these functions exist in a certain nbd. $(x-\delta, x+\delta; y-\delta, y+\delta)$ of the point (x, y) .

Let $(x+h, y+k)$ be any point of this nbd.

$$\text{Then } f(x+h, y+k) - f(x, y) = \{f(x+h, y+k) - f(x, y+k)\} + \{f(x, y+k) - f(x, y)\} \quad (1)$$

The R.H.S. of (1) consists of the sum of two differences.

Consider the first difference $\{f(x+h, y+k) - f(x, y+k)\}$.

Here the second argument of both the terms,

namely $y+k$ are the same.

Therefore we can regard the function $f(x, y+k)$ of one variable x derivable at every point of the interval $(x, x+h)$ with respect to x derivable at every point of the interval $(x, x+h)$ in the nbd. of the point (x, y) .

Hence applying the mean-value theorem, we have

$$f(x+h, y+k) - f(x, y+k) = h f_x(x+\theta, h, y+k)$$

$$\text{Where } 0 < \theta < 1, \theta \text{ being a function of } h \text{ and } k. \quad (2)$$

Since $f_x(x, y)$ is continuous at (x, y) , therefore we can write

$$f_x(x+\theta, h, y+k) - f_x(x, y) = \phi(h, k) \quad (3)$$

$$\text{Where } \phi(h, k) \rightarrow 0 \text{ as } (h, k) \rightarrow (0, 0).$$

Thus from (2) and (3)

$$f(x+h, y+k) - f(x, y+k) = h [f_x(x, y) + \phi(h, k)] \quad (4)$$

Similarly for the second difference on the R.H.S. of (1), we have

$$f(x, y+k) - f(x, y) = k f_y(x, y+\theta_2 k) \quad (5)$$

Where $0 < \theta_2 < 1$: θ_2 , being a function of k
Since $f_x(x, y)$ is continuous at (x, y) , therefore
We can write

$$f_y(x, y + \theta_2 k) - f_y(x, y) = \psi(k)$$

Where $\psi(k) \rightarrow 0$ as $k \rightarrow 0$

Hence from (5)

$$f(x, y+k) - f(x, y) = k [f_x(x, y) + \psi(k)]$$

(6)

This from (1), (4) and (6)

$$f(x+h, y+k) - f(x, y) = h [f_x(x, y) + \phi(h, k)] +$$

$$k [f_y(x, y) + \psi(k)]$$

$$= h f_x(x, y) + k f_y(x, y) + h \phi(h, k) + k \psi(k)$$

Where $\phi(h, k) \rightarrow 0$ as $(h, k) \rightarrow (0, 0)$ and
 $\psi(k) \rightarrow 0$ as $k \rightarrow 0$

Hence ~~$f(x)$~~ $f(x, y)$ is differentiable
at (x, y)

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